THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Solution to Problem Set 2

1. Let $A = (0, 2, 3, 3)$ and $B = (1, -1, 2, 3)$ be two points in $\mathbb{R}^4$. Find the equation of straight line passing through A and B express it in standard form.

   Ans:

   Note that $\overrightarrow{AB} = (1, -3, -1, 0)$ is a direction vector of the required straight line.

   Therefore, the required equation is $(x_1, x_2, x_3, x_4) = (0, 2, 3, 3) + t(1, -3, -1, 0)$, where $t \in \mathbb{R}$.

   By eliminating t, we have

$$x_1 = \frac{x_2 - 2}{-3} = \frac{x_3 - 3}{-1} \qquad \text{and} \qquad x_4 = 3.$$

2. Find the equation of the plane Π containing the straight line

$$L : \frac{x - 4}{2} = \frac{y - 3}{5} = \frac{z + 1}{-2}$$

   and the point $P(2, -4, 2)$.

   Ans:

   Note that $Q(4, 3, -1)$ is a point lying on L and hence on Π, also $\mathbf{a} = (2, 5, -2)$ is a direction vector of L. Therefore, $\mathbf{n} = \overrightarrow{PQ} \times \mathbf{a}$ gives a normal vector of Π and we have $\mathbf{n} = \overrightarrow{PQ} \times \mathbf{a} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$.

   Therefore, the equation of the plane Π is $x - 2y - 4z - 2 = 0$.

3. Find the equation of the straight line given by the intersection of two planes $\Pi_1 : x + y - z = 1$ and $\Pi_2 : x + 2y + 2z = 3$.

   Ans:

   By eliminating x from the two given equations, we have $y + 3z = 2$.

   Let $z = t$, where $t \in \mathbb{R}$. Then, we have $y = 2 - 3z = 2 - 3t$ and $x = 1 - y + z = 1 - (2 - 3t) + t = -1 + 4t$.

   Therefore, the parametric equation of the intersection of Π_1 and Π_2 is given by $(x, y, z) = (-1 + 4t, 2 - 3t, t) = (-1, 2, 0) + t(4, -3, 1)$.

   By eliminating the parameter t, we can get

$$\frac{x + 1}{4} = \frac{y - 2}{-3} = z.$$

4. Let $\Pi : x_1 + 3x_2 - 2x_3 + x_4 + 3 = 0$ be an affine hyperplane and let $P = (7, 21, -7, 3)$ be a point in $\mathbb{R}^4$.

   (a) Find the projection Q of the point P on Π.

   (b) Find the image P' of P under the reflection across Π

   (c) Let $L : (x_1, x_2, x_3, x_4) = (7, 21, -7, 3) + t(3, 10, -4, 4)$ for $t \in \mathbb{R}$, be a straight line passing though P. Find the equation of the straight line L' which is the reflection of L across Π.

   Ans:

(a) Note that $\overrightarrow{PQ}$ is parallel to the normal of Π, so there exists $t \in \mathbb{R}$ such that $\overrightarrow{PQ} = t(1,3,-2,1)$. Therefore, $Q = (7,21,-7,3) + t(1,3,-2,1) = (7+t, 21+3t, -7-2t, 3+t)$.

Since Q lies on Π, we have $(7+t) + 3(21+3t) - 2(-7-2t) + (3+t) + 3 = 90 + 15t = 0$. Then, we have $t = -6$ and $Q = (1,3,5,-3)$.

(b) Note that $\overrightarrow{QP'} = \overrightarrow{PQ} = -6(1,3,-2,1) = (-6,-18,12,-6)$, so we have

$$P' = (1,3,5,-3) + (-6,-18,12,-6) = (-5,-15,17,-9).$$

(c) Let $R = (7,21,-7,3) + t_0(3,10,-4,4) = (7+3t_0, 21+10t_0, -7-4t_0, 3+4t_0)$ be the intersection point of L and Π, where $t_0 \in \mathbb{R}$. Then, $(7+3t_0) + 3(21+10t_0) - 2(-7-4t_0) + (3+4t_0) + 3 = 90 + 45t_0 = 0$.

Therefore, $t_0 = -2$ and $R = (1,1,1,-5)$.

Note that L' passes through R and $\overrightarrow{P'R} = (6,16,-16,4)$ is a direction vector of L'. Therefore, the equation of L' is

$$\frac{x_1 - 1}{6} = \frac{x_2 - 1}{16} = \frac{x_3 - 1}{-16} = \frac{x_4 + 5}{4}$$

or simplified as

$$\frac{x_1 - 1}{3} = \frac{x_2 - 1}{8} = \frac{x_3 - 1}{-8} = \frac{x_4 + 5}{2}.$$

5. Find the equation(s) of the plane(s) Π such that Π is parallel to the plane $\Pi' : x + 2y - 2y + 3 = 0$ and the distance between the origin and Π is 4 units.

Ans:

Since Π is parallel to Π', $(1,2,-2)$ is a normal vector of Π' as well as Π.

Then, the equation of Π is $x + 2y - 2y + D = 0$, where D is a constant.

$$\begin{aligned} \text{The distance between the origin and } \Pi &= 4 \\ \left| \frac{(0) + 2(0) - 2(0) + D}{\sqrt{1^2 + 2^2 + (-2)^2}} \right| &= 4 \\ |D| &= 12 \\ D &= \pm 12 \end{aligned}$$

The equations of required planes are $x + 2y - 2y + 12 = 0$ and $x + 2y - 2y - 12 = 0$.

6. Let $L_1 : \dfrac{x+2}{3} = \dfrac{y-3}{4} = z - 2$ and $L_2 : x - 3 = 5 - y = 1 - z$ be two straight lines in $\mathbb{R}^3$.

(a) Prove that L_1 and L_2 intersect at a point and find the coordinates of that point.

(b) Find the acute angle between L_1 and L_2.

(c) Find the equation of the plane containing L_1 and L_2.

Ans:

(a) Rewrite the equations of L_1 and L_2 in parametric form:

$$\begin{aligned} L_1 : &\quad (x,y,z) = (-2+3s, 3+4s, 2+s) \\ L_2 : &\quad (x,y,z) = (3+t, 5-t, 1-t) \end{aligned}$$

where $s, t \in \mathbb{R}$. Then, we have

$$\begin{aligned} -2 + 3s &= 3 + t \\ 3 + 4s &= 5 - t \\ 2 + s &= 1 - t \end{aligned}$$

By solving the above, we obtain $s = 1$, $t = -2$ and so L_1 and L_2 intersect at $(1,7,3)$.

(b) Note that $\mathbf{a}_1 = (3, 4, 1)$ and $\mathbf{a}_2 = (1, -1, -1)$ are direction vectors of L_1 and L_2 respectively.

Then, the angle between $\mathbf{a}_1$ and $\mathbf{a}_2$ is $\cos^{-1}\left(\frac{\mathbf{a}_1 \cdot \mathbf{a}_2}{|\mathbf{a}_1||\mathbf{a}_2|}\right) = \cos^{-1}\left(\frac{-2}{\sqrt{26}\cdot\sqrt{3}}\right) \approx 103°$.

Therefore, the acute angle between L_1 and L_2 is $180° - 103° = 77°$.

(c) From (b),

$$\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 1 \\ 1 & -1 & -1 \end{vmatrix} = -3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$$

gives a normal of the required plane.

Let the equation of the required plane be $-3x + 4y - 7z + D = 0$. Note that the intersection point $(1, 7, 3)$ of L_1 and L_2 must lie on the plane, so $D = -4$.

Therefore, the equation of the plane containing L_1 and L_2 is $3x - 4y + 7z + 4 = 0$.

7. Let Π be an affine hyperplane in $\mathbb{R}^n$ given by the equation $A_1x_1 + A_2x_2 + \cdots + A_nx_n + B = 0$ and let $P(p_1, p_2, \ldots, p_n)$ be a fixed point.

Show that the perpendicular distance between Π and P is $\left|\frac{A_1p_1 + A_2p_2 + \cdots + A_np_n + B}{\sqrt{A_1^2 + A_2^2 + \cdots + A_n^2}}\right|$.

Ans:

Note that $\mathbf{n} = (A_1, A_2, \ldots, A_n)$ is a normal of Π. Let $Q = (q_1, q_2, \ldots, q_n)$ be a fixed point on Π.

Since Q lies on Π, we have $A_1q_1 + A_2q_2 + \cdots A_nq_n = -B$.

Let θ be the angle between $\vec{n}$ and $\overrightarrow{PQ}$. Then, the perpendicular distance between Π and P

$$= \left||\overrightarrow{PQ}|\cos\theta\right| = \left|\frac{|\overrightarrow{PQ}||\mathbf{n}|\cos\theta}{|\mathbf{n}|}\right| = \left|\frac{\overrightarrow{PQ}\cdot\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{A_1(q_1-p_1) + A_2(q_2-p_2) + \cdots + A_n(q_n-p_n)}{\sqrt{A_1^2 + A_2^2 + \cdots + A_n^2}}\right|$$

$$= \left|\frac{A_1p_1 + A_2p_2 + \cdots + A_np_n + B}{\sqrt{A_1^2 + A_2^2 + \cdots + A_n^2}}\right|$$

(Note: $|-(A_1p_1 + A_2p_2 + \cdots + A_np_n + B)| = |A_1p_1 + A_2p_2 + \cdots + A_np_n + B|$.)

8. Suppose that $\Pi_1 : x + y + z = 1$ and $\Pi_2 : x - y + z = 2$ are two planes in $\mathbb{R}^3$.

(a) Show that the intersection of Π_1 and Π_2 is a straight line and find a parametric equation of that line.

(b) Find the equation(s) of the plane(s) containing all the points which are equidistant from Π_1 and Π_2.

Ans:

(a) We have

$$\begin{cases} x + y + z &= 1 \\ x - y + z &= 2 \end{cases}$$

$$\begin{cases} x + y + z &= 1 \\ -2y &= 1 \end{cases}$$

Then, we have $y = -1/2$. If we put $y = -1/2$ into the first equation, we have $x + z = 3/2$. Let $z = t \in \mathbb{R}$, then $x = 3/2 - t$.

Therefore, $(x, y, z) = (\frac{3}{2} - t, -\frac{1}{2}, t)$ is an intersection point of Π_1 and Π_2 for any $t \in \mathbb{R}$, i.e. Π_1 and Π_2 intersect at a straight line with a parametric equation $(x, y, z) = (\frac{3}{2} - t, -\frac{1}{2}, t)$, where $t \in \mathbb{R}$.

(b) Suppose that $P = (x, y, z)$ is a point in $\mathbb{R}^3$ such that the distance between P and Π_1 and the distance between P and Π_2 are the same. Then, we have

$$\begin{aligned} \left|\frac{x+y+z-1}{\sqrt{1^2+1^2+1^2}}\right| &= \left|\frac{x-y+z-2}{\sqrt{1^2+(-1)^2+1^2}}\right| \\ |x+y+z-1| &= |x-y+z-2| \\ x+y+z-1 &= \pm(x-y+z-2) \end{aligned}$$

The required planes are $2y = -1$ and $2x + 2z = 3$.

9. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a curve defined by $\gamma(t) = (\cos 2t - 1, \sin 2t + 2)$.

(a) Write down an equation of γ in x and y only. What is γ?

(b) Find $\gamma'(t)$.

Ans:

(a) We have $x = \cos 2t - 1$ and $y = \sin 2t + 2$. Then, $x + 1 = \cos 2t$ and $y - 2 = \sin 2t$, so

$$(x+1)^2 + (y-2)^2 = \cos^2 2t + \sin^2 2t = 1.$$

Therefore, γ is the circle centered at $(-1, 2)$ with radius 1.

(b) $\gamma'(t) = (-2\sin 2t, 2\cos 2t)$.

10. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a curve defined by $\gamma(t) = (4\cos 2t, 9\sin 2t)$.

(a) Write down an equation of γ in x and y only. What is γ?

(b) Find $\gamma'(t)$.

Ans:

(a) We have $x = 4\cos 2t$ and $y = 9\sin 2t$. Then, $x/4 = \cos 2t$ and $y/9 = \sin 2t$, so

$$\frac{x^2}{16} + \frac{y^2}{81} = \cos^2 2t + \sin^2 2t = 1.$$

Therefore, γ is the ellipse centered at $(0, 0)$ with semi-major and semi minor axes 9 and 4 respectively.

(b) $\gamma'(t) = (-8\sin 2t, 18\cos 2t)$.

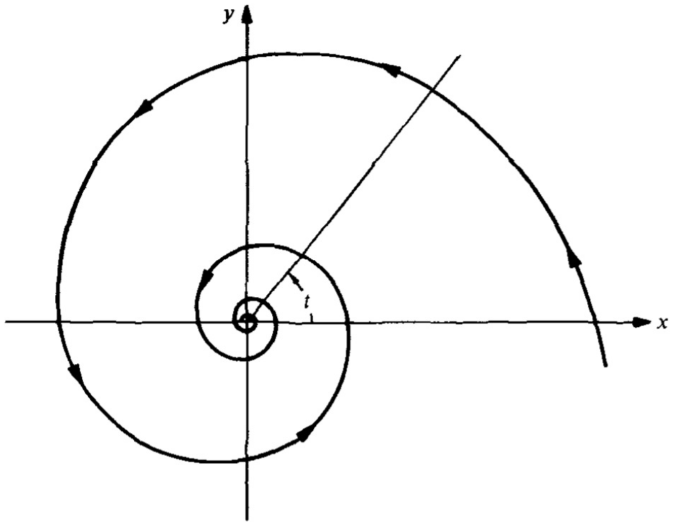
11. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Parametrize the straight line γ which passes through $\mathbf{a}$ and $\mathbf{b}$.

Ans:

Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ defined by $\gamma(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = (1-t)\mathbf{a} + t\mathbf{b}$.

(In particular, we have $\gamma(0) = \mathbf{a}$ and $\gamma(1) = \mathbf{b}$.)

12. Let $\gamma(t) = (ae^{-bt}\cos t, ae^{-bt}\sin t)$ for $t \in \mathbb{R}$, where $a, b > 0$, which is called the *logarithmic spiral*.



(a) Show that as $t \to +\infty$, $\gamma(t)$ approaches the origin.

(b) Show that $\lim_{t\to+\infty} \int_0^t |\gamma'(t)|\,dt$ is finite, that is γ has finite arc length in $[0, +\infty)$.

Ans:

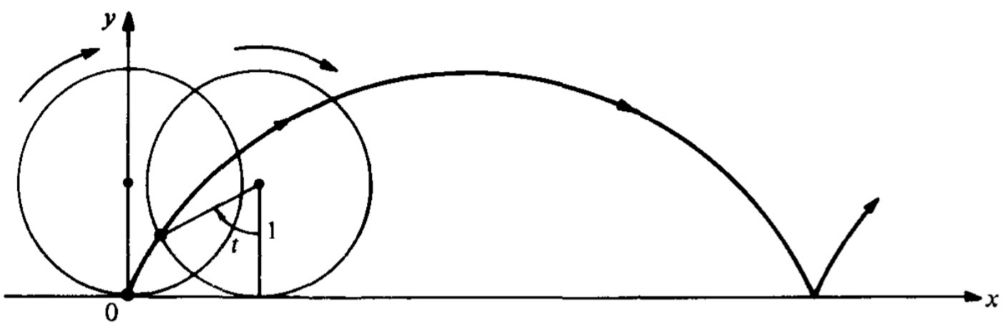
(a) By using sandwich theorem, it can be shown that $\lim_{t\to+\infty} ae^{-bt}\cos t = \lim_{t\to+\infty} ae^{-bt}\sin t = 0$ and the result follows.

(b)

$$\begin{aligned}\int_0^{+\infty} |\gamma'(t)|\,dt &= \int_0^{+\infty} \sqrt{(-abe^{-bt}\cos t - ae^{-bt}\sin t)^2 + (-abe^{-bt}\sin t + ae^{-bt}\cos t)^2}\,dt \\ &= \int_0^{+\infty} ae^{-bt}\sqrt{b^2+1}\,dt \\ &= \frac{a\sqrt{b^2+1}}{b}\end{aligned}$$

which is finite.

13. In the following diagram, a circular disk of radius 1 in the plane xy rolls without slipping along the x-axis and the curve is the locus of a fixed point on the circumference which is called a *cycloid*.



(a) Give a parametrization of the cycloid.

(b) Find the arc length of the cycloid corresponding to a complete rotation of the disk.

Ans:

(a) $\gamma(t) = (t - \sin t, 1 - \cos t)$ for $t \in \mathbb{R}$.

(b)

$$\begin{aligned}
\text{Arc length of the cycloid } \gamma &= \int_0^{2\pi} |\gamma'(t)|\,dt \\
&= \int_0^{2\pi} \sqrt{(1+\cos t)^2 + (\sin t)^2}\,dt \\
&= \int_0^{2\pi} \sqrt{2 + 2\cos t}\,dt \\
&= \int_0^{2\pi} \sqrt{4\sin^2\frac{t}{2}}\,dt \\
&= \int_0^{2\pi} 2\sin\frac{t}{2}\,dt \\
&= 8
\end{aligned}$$